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Equation of motion of a diffusing vortex sheet

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Moore's (1978) equation for following the evolution of a thin layer of uniform vorticity in two dimensions is extended to the case of a non-uniform, instantaneously known, vorticity distribution, using the method of matched asymptotic expansions. In general, the vorticity distribution satisfies a boundary-layer equation. This has a similarity solution in the case of a vortex layer of small thickness in a viscous fluid. Using this solution, an equation of motion of a diffusing vortex sheet is obtained. The equation retains the simplicity of Birkhoff's integro-differential equation for a vortex sheet, while incorporating the effect of viscous diffusion approximately. The equation is used to study the growth of long waves on a Rayleigh layer.

1. Introduction

Many incompressible fluid flows at high Reynolds number are characterized by thin vortex layers, surrounded by irrotational flow. It is usual in such flows to neglect viscosity and replace the vortex layer by a vortex sheet. If the instantaneous position of the vortex sheet can be determined, then the flow can be calculated using the Biot–Savart line integral.

A convenient formulation of the motion of a vortex sheet in an inviscid fluid in two dimensions has been given by Birkhoff (1962). If Γ denotes the net vorticity between one end of the sheet and a point, with complex coordinate $z = \bar{x}+\mathrm{i}\bar{y}$, on the sheet, then the equation of motion can be written in parametric form as $z = \mathscr{Z}(\Gamma, t)$ where $\mathscr{Z}$ satisfies

$$\frac{\partial \mathscr{Z}^*}{\partial t}(\Gamma, t) = -\frac{\mathrm{i}}{2\pi}⨍_0^{\Gamma_e} \frac{\mathrm{d}\Gamma'}{\mathscr{Z}(\Gamma, t)-\mathscr{Z}(\Gamma', t)}; \quad 0 < \Gamma < \Gamma_e, \tag{1.1}$$

where $⨍$ denotes Cauchy principal value integral, an asterisk denotes complex conjugate and Γ_e is the total circulation in the sheet. Equation (1.1) reduces the problem of calculating the position of the vortex sheet to a marching problem in time and would therefore seem suitable for numerical integration. However, chaotic behaviour invariably results, the solutions being sensitive to time-step and discretization procedures used, and successful integration requires use of smoothing techniques. Moore (1981) demonstrated that the chaotic behaviour is a manifestation of a discrete form of the Helmholtz instability of the vortex sheet, whereby the shortest-wave disturbances grow the fastest. Further, a vortex sheet can develop singularities in finite time, suggesting that (1.1) is ill-posed (Saffman 1992, p. 145).

In real flows the short-wave disturbances are suppressed by viscosity which acts to thicken the vortex sheet; for an inviscid model of such a layer, it was shown by Rayleigh (1894) that waves of wavelength less than approximately five times the layer thickness are not amplified. In view of this, Moore (1978) extended the Birkhoff equation to that for a thin vortex layer of uniform vorticity. The equation is valid so long as the radius of curvature of the layer is uniformly large compared with its thickness. It is therefore suitable for considering the evolution of long waves on the layer. However, the fate of short waves, which are quite outside the range of validity of the equation but which are likely to arise in any numerical calculation, need to be considered in order to determine whether chaotic motion would ensue in a numerical integration of the equation. Unfortunately, analysis revealed that while the growth of short waves with length in a certain range would be suppressed by allowing for the thickness of the layer, very short waves would still be spuriously amplified, requiring, once again, the use of smoothing techniques for successful numerical integration of the problem. It is possible that the problem may be overcome by either allowing for a non-uniform distribution of vorticity in the vortex layer or by considering a higher-order correction to Birkhoff's equation.

In this paper we consider the first of these possibilities, namely allowing for a non-uniform distribution of vorticity in the layer. The vorticity is required to decay at least exponentially away from the 'centroid line' C (defined in §2) of the vorticity distribution. The free vortex layer is regarded as a 'double-sided' boundary layer on an evolving space curve C whose equation of motion is sought.

There are two disparate lengthscales, namely the thickness of the layer and its local radius of curvature, in the problem, and the method of matched asymptotic expansions is used, as in Moore (1978), to determine the equation of motion of C when the ratio of these two scales is uniformly small, of $O(\epsilon)$, where $\epsilon \ll 1$. Thus an 'outer' problem based on flow at a large distance from C due to a vortex sheet at C is posed and the solution to this is matched to an 'inner' solution for the flow in the vicinity of C. An equation in terms of an expansion in ϵ is developed.

In §2, the intrinsic coordinate system used is described and the equations of motion are established. In §3, assuming that the vorticity distribution $\bar{\omega}$ is instantaneously known, an equation of motion of the curve C is obtained in terms of $\bar{\omega}$; the details of the matching process are given in Dhanak (1980) and are not reproduced here. The equation retains the simplicity of the vortex sheet model, while incorporating finite-thickness effects approximately. It reduces to the equation given by Moore if $\bar{\omega}$ is taken to be uniform. An interpretation of the equation in terms of forces on the layer is also given in §3. The modified equation involves the momentum thickness δ_2 an equation for which is given in §3, though this cannot be given in closed form. A modification to Kirchoff's invariant for a vortex sheet is also given in §3; it allows for viscous dissipation.

The equation for the non-uniform vorticity layer is used in the Appendix to consider the growth of long waves on a straight layer of steady, non-uniform vorticity in an inviscid fluid and the results are compared with corresponding results of Drazin & Howard (1962) for a mixing layer with a general velocity distribution. The comparison provides a useful check for the equation. In general, the determination of $\bar{\omega}$ requires a solution to the boundary-layer-type equation. However, for the case of an instantaneously created arbitrary vortex sheet undergoing viscous diffusion, the vorticity distribution, valid for small times, can be determined. Thus, in §4, an equation of motion of a diffusing vortex sheet is obtained. The asymptotic equation is valid provided the thickness of the layer of vorticity is small compared to the local radius of curvature. In particular, the asymptotic scheme breaks down in the vicinity of points where the sheet develops curvature singularities; at such points a transition in behaviour from shearing to rotation in a small core is expected (Tryggvasan, Dahm & Skeih 1991). The equation is used in §5 to study the growth of long waves on a Rayleigh layer.

2. Preliminaries

In this section the equations of motion for the flow in the immediate vicinity of the vortex layer are established. The intrinsic coordinate system to be used here was introduced in Moore (1978) and is briefly described below.

Let s be the arc distance measured along a plane curve C (figure 1); C will be identified with the centroid line of the vorticity distribution. Let P' be a point close enough to C for there to be a unique normal from P' to C. Let the normal meet C at P. Then, in a frame $\overline{OXY}$, fixed relative to flow at infinity, the position of P' at time t is given by

$$\boldsymbol{r}(P') = \boldsymbol{R}(s,t) + n\hat{\boldsymbol{n}}(s,t), \tag{2.1}$$

where $\boldsymbol{R}(s,t)$ refers to the point P, n is the distance $\overline{PP'}$ and $\hat{\boldsymbol{n}}(s,t)$ is the unit normal at P; $\hat{\boldsymbol{n}}(s,t)$ points to the left as C is traversed in the increasing s-direction (n is positive in the positive $\hat{\boldsymbol{n}}(s,t)$-direction).

Differentiation of (2.1) and use of Serret–Frenet formulae for a plane curve leads to

$$\mathrm{d}\boldsymbol{r} = \hat{\boldsymbol{s}}\left(1-\frac{n}{\rho}\right)\mathrm{d}s + \hat{\boldsymbol{n}}(s,t)\,\mathrm{d}n, \tag{2.2}$$

where $\rho(s,t)$ is the radius of curvature of C at P and

$$\hat{\boldsymbol{s}} = \frac{\partial \boldsymbol{R}}{\partial s}, \tag{2.3}$$

is the unit tangent vector at P. Hence the coordinate system is orthogonal with line elements $h_1\,\mathrm{d}s$ and $\mathrm{d}n$ where

$$h_1 \equiv h = 1 - n/\rho. \tag{2.4}$$

If $\bar{\boldsymbol{u}}(P')$ is the fluid velocity in the $\overline{OXY}$ frame of reference at P' at time t, then we define relative velocity (u, v) in the (s, n) system by

$$\bar{\boldsymbol{u}}(P') = \frac{\partial \boldsymbol{R}}{\partial t} + n\frac{\partial \hat{\boldsymbol{n}}}{\partial t} + u\hat{\boldsymbol{s}} + v\hat{\boldsymbol{n}}, \tag{2.5}$$

and since

$$\frac{\partial \hat{\boldsymbol{n}}}{\partial t}(s,t) = -\Omega(s,t)\,\hat{\boldsymbol{s}}, \tag{2.6}$$

where $\Omega(s,t)$ is the angular velocity of the coordinate frame $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{n}})$ at a point with fixed s, (2.5) can be written

$$\bar{\boldsymbol{u}}(P') = \left(\hat{\boldsymbol{s}}\cdot\frac{\partial \boldsymbol{R}}{\partial t} - \Omega n + u\right)\hat{\boldsymbol{s}} + \left(\hat{\boldsymbol{n}}\cdot\frac{\partial \boldsymbol{R}}{\partial t} + v\right)\hat{\boldsymbol{n}} \tag{2.7}$$

The continuity equation (see Moore 1978) is given by

$$\frac{\partial u}{\partial s} + \frac{\partial}{\partial n}(hv) = n\frac{\partial \Omega}{\partial s}, \tag{2.8}$$

while the vorticity $\bar{\omega}(s,n,t)$ in the fixed coordinate frame $\overline{OXY}$ is given by

$$\bar{\omega}(s,n,t) = \frac{1}{h}\left(\frac{\partial v}{\partial s} - \frac{\partial}{\partial n}(hu)\right) + 2\Omega. \tag{2.9}$$

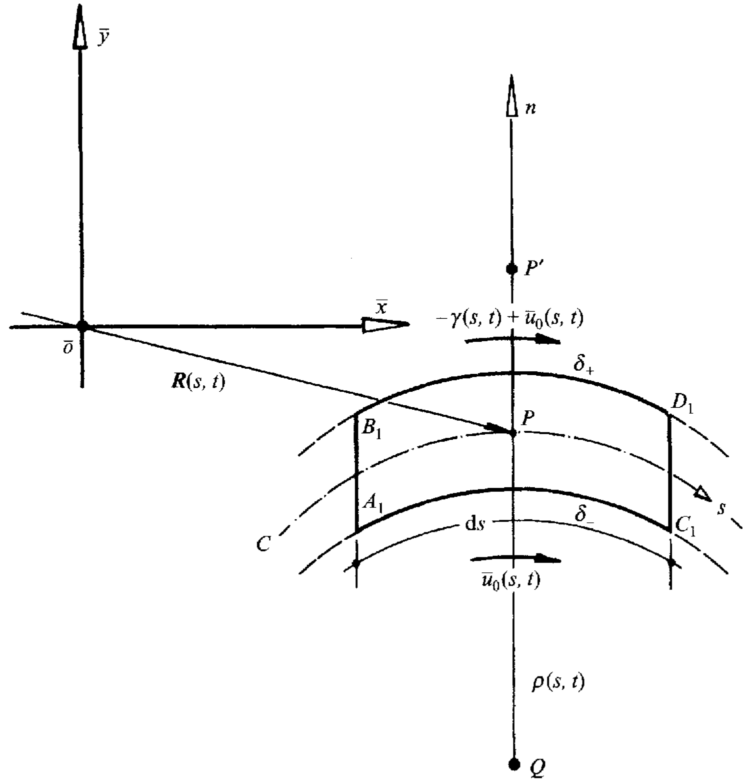

FIGURE 1. An element of a vortex layer with centroid $P(s)$ and radius of curvature $\rho(s)$.

It can be shown that $\bar{\omega}(s,n,t)$ satisfies the equation

$$\frac{\partial \bar{\omega}}{\partial t}+\frac{1}{h}\left(\frac{\partial(\bar{\omega}u)}{\partial s}+\frac{\partial}{\partial n}(h\bar{\omega}v)-\bar{\omega}n\frac{\partial \Omega}{\partial s}\right)=\frac{\bar{\nu}}{h}\left(\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial \bar{\omega}}{\partial s}\right)+\frac{\partial}{\partial n}\left(h\frac{\partial \bar{\omega}}{\partial n}\right)\right), \tag{2.10}$$

where $\bar{\nu}$ is the fluid viscosity. We now define the plane curve $\boldsymbol{r}=\boldsymbol{R}(s,t)$ in the fixed frame $\overline{OXY}$. If we write

$$\bar{\omega}(s,n,t)=\bar{\omega}(\boldsymbol{R}(s,t)+n\hat{\boldsymbol{n}}(s,t),t), \tag{2.11}$$

then $\boldsymbol{R}(s,t)$ is chosen so that uniformly in s and t, $\bar{\omega}$ decays exponentially as $n\rightarrow(\pm)\infty$, and uniformly in s

$$\int \bar{\omega}n\,\mathrm{d}n=0. \tag{2.12}$$

Here the limits of the integral are chosen to be where $\bar{\omega}$ is vanishingly small; Baker & Shelley (1990) have considered an alternative mid-line definition for a layer of uniform vorticity. The choice (2.12) of C ensures that most of the vorticity lies in a thin layer containing C. The advantage of this choice becomes apparent when the circulation of the layer is considered, because the circulation round a curve enclosing the layer between a normal through a point on C characterized by arc distance s and one through $s=0$, at time t, is given by

$$\Gamma(s,t)=\int_0^s\int \bar{\omega}h\,\mathrm{d}n\,\mathrm{d}s. \tag{2.13}$$

That is, on substituting for h from (2.4) and using (2.12),

$$\Gamma(s,t) = \int_0^s \int \bar{\omega}\,\mathrm{d}n\,\mathrm{d}s. \tag{2.14}$$

Hence, the circulation density is

$$\gamma(s,t) = \frac{\partial \Gamma}{\partial s} = \int \bar{\omega}\,\mathrm{d}n, \tag{2.15}$$

independent of the local curvature of the layer. On multiplying the vorticity equation (2.10) by h and integrating across the width of the layer it can be shown that $\gamma(s,t)$ satisfies

$$\frac{\partial \gamma}{\partial t} + \frac{\partial (U_c \gamma)}{\partial s} = \bar{\nu}\frac{\partial}{\partial s}\int \frac{1}{h}\frac{\partial \bar{\omega}}{\partial s}\,\mathrm{d}n, \tag{2.16}$$

where $U_c(s,t)$ is the speed of convection of vorticity in the direction of tangent to the curve C at s and is given by

$$U_c \int \bar{\omega}\,\mathrm{d}n = \int \bar{\omega} u\,\mathrm{d}n. \tag{2.17}$$

A further consequence of the definition (2.12) of the curve C is that on differentiating (2.12) with respect to time and using (2.10) we have the exact invariant relation

$$\int \left\{ \frac{n}{h}\left(\frac{\partial(\bar{\omega}u)}{\partial s} - \bar{\omega} n \frac{\partial \Omega}{\partial s} - \bar{\nu}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial \bar{\omega}}{\partial s}\right)\right) - \frac{\bar{\omega} v}{h} - \frac{\bar{\nu}\bar{\omega}}{\rho h^2} \right\} \mathrm{d}n = 0, \tag{2.18}$$

which for a straight layer reduces to

$$\frac{\partial}{\partial s}\left(\int \bar{\omega} u n\,\mathrm{d}n \right) - \int \bar{\omega} v\,\mathrm{d}n = 0. \tag{2.19}$$

3. Governing equations

We assume that the instantaneous vorticity distribution $\bar{\omega}(s,n,t)$ decays at least as fast as

$$\exp(-|n/\beta_{\pm}|) \quad \text{as} \quad n \to \pm\infty, \tag{3.1}$$

so that the scale of $\beta_{\pm}(s,t)$ gives a measure of the thickness of the vortex layer. To ensure that the layer is thin, we must have, uniformly in s,

$$|\beta_{\pm}(s,t)/\rho(s,t)| \leqq \epsilon, \tag{3.2}$$

where ρ is the radius of curvature of C at position s and $\epsilon \ll 1$. If we regard the jump in velocity, $\gamma(s,t)$, across the layer as an $O(1)$ quantity, then (3.2) implies that $\bar{\omega} = O(\epsilon^{-1})$ uniformly over the length of the sheet (except possibly at the ends of the sheet). In view of the two lengthscales ρ and β, a procedure for a matched asymptotic expansion can be developed in a manner described in Moore (1978) in order to determine the equation for the motion of the centroid curve C in terms of the vorticity distribution. Details of the inner and outer solutions and the matching proccss are lengthy but straightforward and are given in Dhanak (1980). For brevity these will not be reproduced here and only the final results will be given. Basically, in the inner problem, assuming $\bar{\omega}$ to be instantaneously given, we introduce an inner variable $y = \epsilon^{-1} n$ and expand u and v in (2.7) as

$$u = u_0 + \epsilon u_1 + \dots; \quad v = \epsilon v_1 + \epsilon^2 v_2 + \dots \tag{3.3}$$

to obtain, upon substitution in (2.8) and (2.9) and equating coefficients of powers of ϵ to zero, a set of equations for u_0, v_1 and so on; note that v is $O(\epsilon)$. Hence,

$$u_0 = -\int_{-\infty}^{n} \bar{\omega}\, \mathrm{d}n + \bar{u}_0(s,t) + O(\epsilon); \quad v_1 = \int_{-\infty}^{n} \frac{\partial}{\partial s} \Delta \mathrm{d}y - y\bar{u}_0(s,t) + \bar{v}_1(s,t) + O(\epsilon^2), \quad (3.4)$$

with $\bar{u}_0(s,t)$ and $\bar{v}_1(s,t)$ to be determined from matching with the outer solution and Δ defined as in (3.9).

It may be noted that the circulation density equation (2.16) to leading order in ϵ is given by

$$\frac{\partial \gamma}{\partial t} + \frac{\partial (U_c \gamma)}{\partial s} = \bar{\nu} \frac{\partial^2 \gamma}{\partial s^2} + O(\epsilon^2). \quad (3.5)$$

In the outer problem, the flow is characterized by the mean properties of the layer and is insensitive to the actual details of the vorticity distribution. To $O(\epsilon^2)$, the outer solution is obtained by replacing the layer by an equivalent vortex sheet at C. This is a familiar result in boundary-layer theory where the effect of the boundary layer is taken into account by considering the surface to be at a distance equivalent to the displacement thickness δ_1 above the true surface; the error introduced is of $O((\delta_1/l)^2)$ where l is a typical length.

Let $Z(s,t)$ be the complex parametric equation of the centroid line C, where

$$Z(s,t) = \boldsymbol{R}(s,t)\cdot\hat{\boldsymbol{x}} + \mathrm{i}\boldsymbol{R}(s,t)\cdot\hat{\boldsymbol{y}}, \quad (3.6)$$

with $\boldsymbol{R}(s,t)$ as in (2.1) so that a point $z = \bar{x} + \mathrm{i}\bar{y}$ in the vicinity of the curve may be written

$$z = Z(s,t) + \mathrm{i}n\,\mathrm{e}^{\mathrm{i}\alpha}, \quad (3.7)$$

where $\alpha(s,t)$ is the inclination of the tangent $\hat{s}(s,t)$ to the $\overline{OX}$-direction. For each z there is a unique α provided that $|n| \ll \rho(s,t)$. Then, by matching the inner solution with the outer solution, it can be shown that

$$\frac{\partial Z^*}{\partial t} + U_c \mathrm{e}^{-\mathrm{i}\alpha} = -\frac{\mathrm{i}}{2\pi}\int_0^{a_0(t)} \frac{\gamma(s',t)\,\mathrm{d}s'}{Z(s,t)-Z(s',t)} - \frac{\mathrm{e}^{-\mathrm{i}\alpha}}{\gamma}\left[\frac{1}{\rho}\gamma^2\delta_2 + \mathrm{i}\frac{\partial(\gamma^2\delta_2)}{\partial s}\right] + O(\epsilon^2), \quad (3.8)$$

where an asterisk denotes complex conjugate, $a_0(t)$ denotes the arc distance between the two ends of C and

$$\delta_2 = \gamma^{-2}\int_{-\infty}^{\infty} \Delta(\gamma - \Delta)\,\mathrm{d}n; \quad \Delta(s,n,t) = \int_{-\infty}^{n} \bar{\omega}(s,n',t)\,\mathrm{d}n'. \quad (3.9)$$

Here, $-\Delta(s,n,t)$ is, to $O(\epsilon^2)$, the jump in the tangential velocity at station s between position n and $n = -\infty$ so that $\Delta(s,\infty,t) = \gamma(s,t)$, the circulation density; the $O(\epsilon)$ correction is zero at an end where $\gamma = 0$. δ_2 can be identified with the momentum thickness of the layer if we define the momentum thickness as

$$\delta_2 = \int_{-\infty}^{\infty} \frac{(U_2 - u)(U_1 - u)}{(U_1 - U_2)^2}\,\mathrm{d}n,$$

where U_1 and U_2 are respectively the 'free-stream' tangential velocities at $n = \pm\infty$ respectively and u is the local component of velocity tangential to C.

We now introduce Birkhoff's circulation coordinate Γ. Suppose $s(\Gamma, t)$ is the arc distance along C to a normal section which has constant net vorticity Γ between it and the end $s = 0$ for $t \geqslant 0$. Then

$$\Gamma(s,t) = \int_0^s \gamma(s',t)\,\mathrm{d}s' \tag{3.10}$$

so that
$$0 = \gamma(s,t)\left(\frac{\partial s}{\partial t}\right)_\Gamma + \int_0^s \frac{\partial \gamma(s',t)}{\partial t}\,\mathrm{d}s'.$$

On using the circulation density equation (3.5) and noting that $\gamma(s,t) \neq 0$ for $s \neq 0$, $a(t)$, we have

$$\left(\frac{\partial s}{\partial t}\right)_\Gamma = U_c - \frac{\bar{\nu}}{\gamma}\frac{\partial \gamma}{\partial s}\frac{\partial}{\partial s}, \tag{3.11}$$

so that if Γ, n, t are regarded as the new independent variables and we write

$$\mathscr{Z}(\Gamma,t) = Z(s,t); \quad U(\Gamma,t) = \gamma(s,t); \quad \hat{\Delta}(\Gamma,n,t) = \Delta(s,n,t) \quad \text{and} \quad \hat{\delta}_2(\Gamma,t) = \delta_2(s,t); \tag{3.12}$$

then
$$\left(\frac{\partial}{\partial t}\right)_\Gamma = \frac{\partial}{\partial t} + U_c\frac{\partial}{\partial s}.$$

Further, we note that

$$\frac{\partial \alpha}{\partial s} = \frac{1}{\rho}; \quad \frac{\partial Z}{\partial s} = \mathrm{e}^{\mathrm{i}\alpha} \quad \text{and} \quad \frac{\partial}{\partial s} = U\frac{\partial}{\partial \Gamma}, \tag{3.13}$$

so that
$$\mathrm{e}^{\mathrm{i}\alpha} = U\frac{\partial \mathscr{Z}}{\partial \Gamma} \quad \text{and hence} \quad U^2 = \left\{\frac{\partial \mathscr{Z}}{\partial \Gamma}\frac{\partial \mathscr{Z}^*}{\partial \Gamma}\right\}^{-1}. \tag{3.14}$$

Thus it can be shown that (3.7) can be written

$$\frac{\partial \mathscr{Z}^*(\Gamma,t)}{\partial t} = -\frac{\mathrm{i}}{2\pi}\oint_0^{\Gamma_e} \frac{\mathrm{d}\Gamma'}{\mathscr{Z}(\Gamma,t)-\mathscr{Z}(\Gamma',t)} - \mathrm{i}\frac{\partial}{\partial \Gamma}\left[\hat{\delta}_2\, U^3\frac{\partial \mathscr{Z}^*}{\partial \Gamma}\right] - \bar{\nu}U\frac{\partial U}{\partial \Gamma}\frac{\partial \mathscr{Z}^*}{\partial \Gamma} + O(\epsilon^2). \tag{3.15}$$

Then from (3.13) it can be shown that

$$\frac{\partial U}{\partial t} = -U^3\,\mathrm{Re}\left(\frac{\partial \mathscr{Z}}{\partial \Gamma}\left(\frac{\partial \mathscr{I}}{\partial \Gamma} - \bar{\nu}U\frac{\partial U}{\partial \Gamma}\frac{\partial \mathscr{Z}^*}{\partial \Gamma}\right)\right) + O(\epsilon). \tag{3.16}$$

where Re denotes real part and

$$\mathscr{I} = -\frac{\mathrm{i}}{2\pi}\int_0^{\Gamma_e}\frac{\mathrm{d}\Gamma'}{\mathscr{Z}(\Gamma,t)-\mathscr{Z}(\Gamma',t)}.$$

If $\bar{\nu} = 0$ and all the vorticity were concentrated in a sheet at C, all terms except the first one on the right-hand side of (3.15) vanish and we recover Birkhoff's equation for the motion of a vortex sheet. Further, if the vorticity $\bar{\omega} = \bar{\omega}_0$, a constant for $-H(\Gamma,t) < n < H(\Gamma,t)$ and $\bar{\omega} = 0$ otherwise with $\bar{\nu} = 0$, then

$$\hat{\Delta} = \bar{\omega}_0(n+H) \quad \text{and} \quad H = U/(2\bar{\omega}_0) \tag{3.17}$$

and on substituting this into (3.15) and (3.16), we recover Moore's (1978) equation for the motion of a thin layer of uniform vorticity. It may be noted, however that if

$\bar{\nu} \neq 0$ and the layer is approximated by a vortex sheet of zero thickness at C, then a modified Birkhoff's equation results:

$$\frac{\partial \mathscr{Z}^*(\Gamma, t)}{\partial t} = -\frac{\mathrm{i}}{2\pi}\int_0^{\Gamma_e} \frac{\mathrm{d}\Gamma'}{\mathscr{Z}(\Gamma, t) - \mathscr{Z}(\Gamma', t)} - \bar{\nu} U \frac{\partial U}{\partial \Gamma}\frac{\partial \mathscr{Z}^*}{\partial \Gamma}. \tag{3.18}$$

In (3.15) we have retained the third term on the right-hand side on the assumption that $\epsilon^2 Vl/\bar{\nu} = o(1)$, where V and l are velocity and length scales associated with the motion of the curve C. If, however, $\epsilon^2 Vl/\bar{\nu} = O(1)$, then the third term is of the same order as the error term and, for consistency, cannot be retained. Thus, in this case, (3.15) becomes

$$\frac{\partial \mathscr{Z}^*(\Gamma, t)}{\partial t} = -\frac{\mathrm{i}}{2\pi}\int_0^{\Gamma_e} \frac{\mathrm{d}\Gamma'}{\mathscr{Z}(\Gamma, t) - \mathscr{Z}(\Gamma', t)} - \mathrm{i}\frac{\partial}{\partial \Gamma}\left[\hat{\delta}_2 U^3 \frac{\partial \mathscr{Z}^*}{\partial \Gamma}\right] + O(\epsilon^2). \tag{3.19}$$

Further, using the vorticity equation to leading order in ϵ, it can be shown (Dhanak 1980) that in this case $\hat{\delta}_2$ satisfies the following energy equation:

$$U^2 \frac{\partial}{\partial t}\left(\frac{\hat{\delta}_2}{U}\right) + \frac{1}{2}\frac{\partial}{\partial \Gamma}(U^3 \hat{\delta}_2) = \frac{\partial}{\partial \Gamma}(U^2 \hat{\delta}_3) + \frac{2\bar{\nu}}{U}\int_{-\infty}^{\infty} \bar{\omega}^2 \,\mathrm{d}n, \tag{3.20}$$

where $\hat{\delta}_3$ is given by

$$\hat{\delta}_3 = \int_{-\infty}^{\infty} \frac{\hat{\Delta}^2}{U^2}\left(1 - \frac{\hat{\Delta}}{U}\right) \mathrm{d}n. \tag{3.21}$$

$\hat{\delta}_3$ corresponds to the energy thickness of the boundary-layer theory; a similar equation may be obtained for the case $\epsilon^2 Vl/\bar{\nu} = o(1)$. Thus, in general, the equations of motion are not in closed form and in order to determine $\hat{\delta}_2$ the instantaneous vorticity distribution, governed by the leading-order approximation to (2.10), needs to be established. In §4, we consider the case for which the non-uniform vorticity distribution can be obtained approximately in the two cases $\epsilon^2 Vl/\bar{\nu} = o(1)$ and $\epsilon^2 Vl/\bar{\nu} = O(1)$, while in the Appendix, we use (3.15) to consider the stability of a straight steady inviscid ($\bar{\nu} = 0$) mixing layer with a general velocity distribution and show that the growth rates of disturbances to the layer, obtained using (3.14), are in agreement with Drazin & Howard (1962).

A useful invariant used in numerical calculations involving vortex sheets to check for numerical accuracy is Kirchoff's invariant function for a vortex sheet, given by

$$\tilde{W}_0 = -\frac{\rho_0}{8\pi}\int_0^{\Gamma_e}\int_0^{\Gamma_e} \log|\mathscr{Z}(\Gamma, t) - \mathscr{Z}(\Gamma', t)| \,\mathrm{d}\Gamma \,\mathrm{d}\Gamma', \tag{3.22}$$

where ρ_0 is the fluid density. Using (3.19) and (3.20), it can be shown that a modification to this invariant for a thin layer of non-uniform vorticity in the case $\epsilon^2 Vl/\bar{\nu} = O(1)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{W}_0 - \frac{\rho_0}{2}\int_0^{\Gamma_e} U\delta_2 \,\mathrm{d}\Gamma\right) = -\bar{\mu}\int_0^{\Gamma_e} \frac{1}{U}\int_{-\infty}^{\infty} \bar{\omega}^2 \,\mathrm{d}n \,\mathrm{d}\Gamma + O(\epsilon^2), \tag{3.23}$$

where $\bar{\mu} = \bar{\nu}\rho_0$; a similar result may be obtained in the case $\epsilon^2 Vl/\bar{\nu} = o(1)$. Hence an invariance to $O(\epsilon^2)$ is obtained if $\bar{\nu} = 0$.

Interpretation of the modified Birkhoff equation (3.15)

A simple interpretation of the modified Birkhoff equation (3.15) may be given as follows; a similar interpretation for Moore's (1978) equation is given in Saffman (1992, pp. 160–163).

Consider, at a fixed instant, an element of length $\mathrm{d}s$ of the layer with local radius of curvature $\rho(s,t)$ and the local centre of curvature at $Q(s,t)$; s being identified with the arc distance measured along the line joining the centroid of each element. Let $P(s,t)$ be the instantaneous centroid of the vorticity distribution $\bar{\omega}(s,n,t)$ in the element, n being distance measured from P along $\overline{QP}$. The surfaces A_1C_1 and B_1D_1 (see figure 1) are chosen so that the vorticity there is effectively zero; this is always possible in view of the exponential decay of vorticity (cf. (3.1)) as $\overline{QP}$ is traversed away from P. Let the instantaneous distance from P to A_1C_1, and from P to B_1D_1 measured along $\overline{QP}$ be $\delta_{\mp}(s,t)$ respectively.

The jump in the tangential velocity across the element is given by the circulation density $-\gamma(s,t)$. Then, for flow at great distances from P, the element can be represented by a point vortex of strength $\gamma\,\mathrm{d}s$ at P. If in a coordinate frame fixed with respect to flow at infinity the position of P is given by $\boldsymbol{r} = \boldsymbol{R}(s,t)$, then in the absence of any distribution of vorticity (i.e. if we have a vortex sheet at C),

$$\frac{\partial \boldsymbol{R}(s,t)}{\partial t} = \boldsymbol{V}_0 \equiv (U_0, V_0), \tag{3.24}$$

where $\boldsymbol{V}_0$ is the velocity induced by the other elements of the layer and includes the velocity with which P is convected along the sheet. Equation (3.24) is essentially equivalent to Birkhoff's equation (1.1).

When the vorticity is distributed in a layer, the velocity contribution from all the other elements of the layer to the element at s may be regarded as due to an approximate vortex sheet lying along C but excluding the section coinciding with the element at s. Then the distribution of vorticity in the element at s will give rise to an extra velocity at $P(s)$, given by

$$(U_1, V_1) = \frac{\partial \boldsymbol{R}(s)}{\partial t} - (U_0, V_0). \tag{3.25}$$

The existence of this extra velocity at the vortical element P produces an extra force on the element given by the Kutta lift and the difference between the local pressure gradient and viscous forces associated with the element and those associated with a corresponding element enclosing an equivalent vortex sheet of zero thickness along C as

$$\gamma\,\mathrm{d}s(U_1, V_1)\times\boldsymbol{k} + \mathrm{d}s\left(\int_{\delta_-}^{\delta_+}\left[-\frac{\partial p}{\partial s} + \bar{\nu}\frac{\partial^2 u}{\partial s^2}\right]\mathrm{d}n, 0\right) - \mathrm{d}s\left(\left[n\left(-\frac{\partial p}{\partial s} + \bar{\nu}\frac{\partial^2 u}{\partial s^2}\right)\right]_{\delta_-}^{\delta_+}, 0\right) \equiv (F, G), \tag{3.26}$$

where $\boldsymbol{k}$ is a unit vector normal to the plane of flow and (u, v) is fluid velocity in the neighbourhood of P relative to (U_0, V_0) with u given by (3.4) ($u = \bar{u}_0(s,t)$ at $n = \delta_-$, and $u = -\gamma(s,t) + \bar{u}_0(s,t)$ at $n = \delta_+$); the second term on the left is the contribution from the vortex layer while the third term is the contribution from a corresponding element enclosing an equivalent sheet of zero thickness. Here, we have assumed that the fluid has unit density and the pressure p remains sensibly constant across the layer. In fact, to leading order, $p(s, \delta_{\pm}, t)$ has the same value for the layer as for the corresponding sheet so that the pressure terms in (3.26) cancel. Further, using (3.4), it can be shown that the viscous terms in (3.26) also cancel. Then, (F, G) is given by the difference between the rate of change of momentum in the layer element and in the corresponding element associated with the vortex sheet flow. (F, G) is determined below. Meanwhile, it may be noted that the leading-order constancy of the pressure across the layer implies that

$$-\frac{\partial p}{\partial s}=\frac{\partial \bar{u}_0}{\partial t}+\frac{1}{2}\frac{\partial \bar{u}_0^2}{\partial s}-\bar{\nu}\frac{\partial^2 \bar{u}_0}{\partial s^2}=\frac{\partial}{\partial t}(-\gamma+\bar{u}_0)+\frac{1}{2}\frac{\partial}{\partial s}(-\gamma+\bar{u}_0)^2-\bar{\nu}\frac{\partial^2}{\partial s^2}(-\gamma+\bar{u}_0), \quad (3.27)$$

so that

$$\frac{\partial \gamma}{\partial t}+\frac{\partial}{\partial s}(\gamma(-\tfrac{1}{2}\gamma+\bar{u}_0))=\bar{\nu}\frac{\partial^2 \gamma}{\partial s^2} \quad (3.28)$$

which is the circulation density equation (3.5) with $U_c=-\frac{1}{2}\gamma+\bar{u}_0$.

The difference between the rate of change of momentum in the tangential direction in the layer element and in the corresponding element associated with the vortex sheet flow is to leading order given by

$$F=\int_{\delta_-}^{\delta_+}\left\{\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial s}+\frac{\partial (uv)}{\partial n}\right\}\mathrm{d}n-\left[n\left(\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial s}\right)+uv\right]_{\delta_-}^{\delta_+}. \quad (3.29)$$

Using (3.4) and integrating by parts it can be shown that

$$F=-\frac{\partial}{\partial s}\int_{-\infty}^{\infty}\Delta(\gamma-\Delta)\,\mathrm{d}n. \quad (3.30)$$

Hence, from (3.26)

$$V_1=-\frac{1}{\gamma}\frac{\partial}{\partial s}\int_{-\infty}^{\infty}\Delta(\gamma-\Delta)\,\mathrm{d}n. \quad (3.31)$$

The difference between the rate of change of momentum in the normal direction in the layer element and in the corresponding element associated with the vortex sheet flow is, to leading order, given by

$$G=\int_{\delta_-}^{\delta_+}\frac{u^2}{\rho}\mathrm{d}n-\left[n\frac{u^2}{\rho}\right]_{\delta_-}^{\delta_+}, \quad (3.32)$$

representing the net change in centrifugal acceleration. Using (3.4) and integrating by parts it can be shown that

$$G=-\frac{\mathrm{d}s}{\rho}\int_{-\infty}^{\infty}\Delta(\gamma-\Delta)\,\mathrm{d}n. \quad (3.33)$$

Hence, from (3.26)

$$U_1=\frac{1}{\gamma\rho}\int_{-\infty}^{\infty}\Delta(\gamma-\Delta)\,\mathrm{d}n. \quad (3.34)$$

Both (3.31) and (3.34) are in agreement with the corresponding terms in (3.8).

4. Diffusing viscous vortex sheet

In general, (2.10) needs to be solved to determine the instantaneous vorticity distribution and hence evaluate δ_2. In the case of an instantaneously created arbitrary vortex sheet in flow at high Reynolds number, it is evident that two timescales are involved: the e-folding time associated with the evolution of the curve C and a slower, viscous diffusion timescale. Since the layer thickness parameter ϵ is necessarily a function of the Reynolds number, it is appropriate to consider non-dimensionalized equations. For a typical lengthscale l, a velocity scale V and an e-folding timescale

l^2/V, we introduce a viscous (non-dimensional) time $T = \epsilon^{-2}t$. Then in terms of the inner variables $x = s$ and $y = \epsilon^{-1}n$, with the vorticity $\bar{\omega}(s,n,t) = \omega(x,y,t,T)$, the vorticity equation (2.10) can be expressed in non-dimensional form as

$$\frac{\partial \omega}{\partial t}+\frac{1}{\epsilon^2}\frac{\partial \omega}{\partial T}+\frac{1}{h}\left(\frac{\partial(\omega u)}{\partial x}+\frac{\partial}{\partial y}(h\omega v)-\epsilon\omega y\frac{\partial \Omega}{\partial x}\right)=\frac{1}{hR}\left(\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \omega}{\partial x}\right)+\frac{1}{\epsilon^2}\frac{\partial}{\partial y}\left(h\frac{\partial \omega}{\partial y}\right)\right), \quad (4.1)$$

where $R = Vl/\bar{\nu}$ and $h = (1-\epsilon y/\rho)$. We further expand $\omega(x,y,t,T)$ as

$$\omega = \epsilon^{-1}\omega_{-1}+\omega_0+\ldots, \quad (4.2)$$

with $\int \omega_{-1}\,dy = \gamma(x,t), \int \omega_m\,dy = 0$ $(m > -1)$. For determining $O(\epsilon)$ terms in (3.15), only an $O(\epsilon^{-1})$ approximation to ω is required. We consider two possibilities:

(i) $\epsilon = o(R^{-\frac{1}{2}})$

In this case, on equating $O(\epsilon^{-3})$ terms to zero in (4.1), after substituting for u, v and ω, we obtain

$$\frac{\partial \omega_{-1}}{\partial T} = \frac{1}{R}\frac{\partial^2 \omega_{-1}}{\partial y^2}, \quad (4.3)$$

which has the required solution

$$\omega_{-1} = \frac{\gamma R^{\frac{1}{2}} e^{-\eta^2}}{2(\pi T)^{\frac{1}{2}}}; \quad \eta = \frac{R^{\frac{1}{2}} y}{2T^{\frac{1}{2}}}, \quad (4.4)$$

so that, in view of (4.2), the thickness $\delta_2(s,t)$ in (3.9), non-dimensionalized with respect to l, is given by

$$\delta_2(x,T) = \epsilon\left(\frac{2T}{\pi R}\right)^{\frac{1}{2}} + O(\epsilon^2). \quad (4.5)$$

Hence, (3.15) in non-dimensionalized form becomes, on re-introducing t in (4.5),

$$\frac{\partial \mathscr{Z}^*(\Gamma,t)}{\partial t} = -\frac{i}{2\pi}\int_0^1 \frac{d\Gamma'}{\mathscr{Z}(\Gamma,t)-\mathscr{Z}(\Gamma',t)} - i\left(\frac{2t}{\pi R}\right)^{\frac{1}{2}}\frac{\partial}{\partial \Gamma}\left[U^3\frac{\partial \mathscr{Z}^*}{\partial \Gamma}\right] - \frac{U}{R}\frac{\partial U}{\partial \Gamma}\frac{\partial \mathscr{Z}^*}{\partial \Gamma} + O\left(\frac{t}{R}\right);$$

$$t = o(1), \quad (4.6)$$

assuming that $U(\partial U/\partial \Gamma)(\partial \mathscr{Z}^*/\partial \Gamma)$ is $O(1)$.

(ii) $\epsilon = R^{-\frac{1}{2}}$

For this case, which essentially corresponds to $t = O(1)$ for an instantaneously created sheet, we have from (4.1) that ω is independent of T and ω_{-1} satisfies

$$\frac{\partial \omega_{-1}}{\partial t}+\frac{\partial(\omega_{-1}u_0)}{\partial x}+\frac{\partial(\omega_{-1}v_1)}{\partial y} = \frac{\partial^2 \omega_{-1}}{\partial y^2}. \quad (4.7)$$

Since u_0 and v_1 depend on ω_{-1}, this is a nonlinear equation and would, in general, require a numerical treatment. Here, we develop a series approximation as is done in boundary-layer theory (Goldstein 1938, pp. 183–184). Thus we write

$$\omega_{-1} = \gamma(x,t)\,\hat{\omega}_{-1}, \quad (4.8)$$

so that it follows from (4.7) that

$$\frac{\partial \hat{\omega}_{-1}}{\partial t}-\frac{\partial^2 \hat{\omega}_{-1}}{\partial y^2} = -\frac{1}{\gamma}\left[\frac{\partial \gamma}{\partial t}\hat{\omega}_{-1}+\frac{\partial(\omega_{-1}u_0)}{\partial x}+\frac{\partial(\omega_{-1}v_1)}{\partial y}\right]. \quad (4.9)$$

The first approximation is obtained by setting the right-hand side of (4.9) to zero and using $\int_{-\infty}^{\infty} \hat{\omega}_{-1}\,\mathrm{d}y = 1$; this is given by (4.4) with $T = Rt$. The second approximation is then obtained by substituting for $\hat{\omega}_{-1}$ on the right-hand side of (4.9) and solving the resulting non-homogenous equation and applying the appropriate boundary conditions. Hence,

$$\omega_{-1} \approx \frac{\gamma \mathrm{e}^{-\eta^2}}{2(\pi t)^{\frac{1}{2}}} + \frac{t}{2(\pi t)^{\frac{1}{2}}}\frac{\partial \gamma}{\partial x}\left\{2\pi^{\frac{1}{2}}\eta\,\mathrm{erf}\,\eta(1-\mathrm{erf}\,\eta) - \mathrm{e}^{-\eta^2}\left[\frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\eta + \tfrac{1}{2}(\eta^2-\tfrac{1}{2}) - \frac{\eta\,\mathrm{e}^{-\eta^2}}{2\pi^{\frac{1}{2}}}\right.\right.$$

$$\left.\left. -(\eta^2-\tfrac{1}{2})\,\mathrm{erf}\,\eta\right]\right\} + \frac{t}{2Ut^{\frac{1}{2}}}\left(\frac{\partial \gamma}{\partial t} + U_c\frac{\partial \gamma}{\partial x}\right)(\eta^2-\tfrac{1}{2})\,\mathrm{e}^{-\eta^2}; \quad \eta = \frac{n}{2t^{\frac{1}{2}}}. \tag{4.10}$$

Thus in this case, in view of (4.2), the thickness $\delta_2(s,t)$ in (3.9), non-dimensionalized with respect to l, upon introduction of U and variable Γ in (4.10), is given by

$$\delta_2(\Gamma,t) = \left(\frac{2t}{\pi R}\right)^{\frac{1}{2}}\left(1 + \frac{t}{2U}\frac{\partial U}{\partial t} + O(t^2)\right) + O\left(\frac{1}{R}\right). \tag{4.11}$$

Then the non-dimensional form of (3.19) is given by

$$\frac{\partial \mathscr{Z}^*(\Gamma,t)}{\partial t} = \mathscr{I} - \mathrm{i}\left(\frac{2t}{\pi R}\right)^{\frac{1}{2}}\left\{\frac{\partial}{\partial \Gamma}\left[U^3\frac{\partial \mathscr{Z}^*}{\partial \Gamma}\left(1 - tU^2\,\mathrm{Re}\left(\frac{\partial \mathscr{Z}}{\partial \Gamma}\frac{\partial \mathscr{I}}{\partial \Gamma}\right)\right)\right] + O(t^2)\right\} + O\left(\frac{1}{R}\right);$$

$$t = O(1), \tag{4.12}$$

where $\mathscr{I}$ is as in (3.16). It may be noted that the $O(1/R)$ retained in (4.6) is now of the same order as the error term and is therefore, for consistency, amalgamated in the latter.

Equations (4.6) and (4.12) are the main results of this section.

5. Growth of long waves on a Rayleigh layer

Suppose that, instantaneously at $t = 0$, an infinitely long straight vortex sheet of unit strength is created in a fluid so that if undisturbed its configuration would be given by the non-dimensional equation $\mathscr{Z}(\Gamma,t) = \Gamma$. Suppose now that at $t = 0$ the sheet is so disturbed as to assume an instantaneous shape

$$\mathscr{Z} = \Gamma + f(\Gamma,t), \tag{5.1}$$

where $|\partial f/\partial \Gamma| \ll 1$. Then, as in §4, we consider two cases:

(i) $\epsilon = o(R^{-\frac{1}{2}})$

If (5.1) is substituted into (4.6) and only terms linear in f are retained, then after a little algebra, we obtain

$$\frac{\partial f^*}{\partial t} = \mathscr{I}_1 + \mathrm{i}\left(\frac{2t}{\pi R}\right)^{\frac{1}{2}}\left(\frac{3}{2}\frac{\partial^2 f}{\partial \Gamma^2} + \frac{1}{2}\frac{\partial^2 f^*}{\partial \Gamma^2}\right) + \frac{1}{2R}\left(\frac{\partial^2 f}{\partial \Gamma^2} + \frac{\partial^2 f^*}{\partial \Gamma^2}\right), \tag{5.2}$$

where
$$\mathscr{I}_1 = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \frac{f(\Gamma,t) - f(\Gamma',t)}{(\Gamma-\Gamma')^2}\,\mathrm{d}\Gamma'. \tag{5.3}$$

Here, $f(\Gamma,t)$ is chosen so that it represents a sinusoidal disturbance of spatial wavenumber k (non-dimensionalized), i.e.

$$f(\Gamma,t) = a(t)\,\mathrm{e}^{\mathrm{i}k\Gamma} + b(t)\,\mathrm{e}^{-\mathrm{i}k\Gamma}, \tag{5.4}$$

where $a(t)$ and $b(t)$ are complex-valued functions which need to be determined. On substituting (5.4) into (5.2) and evaluating $\mathscr{I}_1$ by contour integration, terms proportional to $e^{\mp ik\Gamma}$ can be equated to give

$$\frac{da^*}{dt} = \frac{ik}{2}\left(b - k\left(\frac{2t}{\pi R}\right)^{\frac{1}{2}}[3b+a^*] + \frac{ik}{R}[b+a^*]\right), \tag{5.5a}$$

$$\frac{db^*}{dt} = \frac{ik}{2}\left(a - k\left(\frac{2t}{\pi R}\right)^{\frac{1}{2}}[3a+b^*] + \frac{ik}{R}[a+b^*]\right). \tag{5.5b}$$

The form of (5.5) suggests that a possible choice of solution is

$$a(t) \equiv b(t) \equiv \alpha(t) + i\beta(t), \tag{5.6}$$

say. This choice is not unique (e.g. $a \equiv -b$ is also a solution) but it will suffice for our purpose.

Putting (5.6) into (5.5) and equating real and imaginary parts gives

$$\dot{\alpha} = -\tfrac{1}{2}k(1 - 2k(2t/\pi R)^{\frac{1}{2}})\,\beta - \frac{k^2}{R}\alpha, \tag{5.7a}$$

$$\dot{\beta} = -\tfrac{1}{2}k(1 - 4k(2t/\pi R)^{\frac{1}{2}})\,\alpha, \tag{5.7b}$$

where the dot denotes differentiation with respect to time. If we consider quasi-steady solutions to (5.7) in which $\delta_2 = (2t/\pi R)^{\frac{1}{2}}$ is considered to be sensibly constant, then these are of the form $e^{\sigma t}$ where the growth rate σ is given by

$$\sigma^2 + \frac{k^2}{R}\sigma - \tfrac{1}{4}k^2(1-2k\delta_2)(1-4k\delta_2) = 0. \tag{5.8}$$

For a finite Reynolds number but negligible thickness, the amplifying solution has a growth rate

$$\sigma = \frac{1}{2}\left[\left(\frac{k^4}{R^2} + k^2\right)^{\frac{1}{2}} - \frac{k^2}{R}\right], \tag{5.9}$$

which is a modification of the corresponding inviscid flow result associated with Helmholtz instability. It may be noted that as $k \to \infty$, $\sigma \to \frac{1}{4}R$, unlike in the case of Helmholtz instability where $\sigma \to \infty$ in the limit. For a finite value of $\delta_2 \neq 0$ with $k\delta_2 \ll 1$, $R \gg 1$, the amplifying solution is given by

$$\sigma = \tfrac{1}{2}k(1 - 3k\delta_2 + O(k^2\delta_2^2)). \tag{5.10}$$

This may be compared with the corresponding limit of Rayleigh's (1894) result for the growth rate of disturbances on an equivalent layer of constant vorticity of thickness h; the two results are identical if we choose $h = \frac{9}{2}\delta_2$.

(ii) $\epsilon = R^{-\frac{1}{2}}$

In this case the disturbance equations corresponding to (5.7) become, using (4.12),

$$\dot{\alpha}_1 = -(1 - 4(t_1/R_1)^{\frac{1}{2}})\,\beta_1, \tag{5.11a}$$

$$\dot{\beta}_1 = -(1 - 8(t_1/R_1)^{\frac{1}{2}})\,\alpha_1 - 2t_1(t_1/R_1)^{\frac{1}{2}}\beta_1, \tag{5.11b}$$

where $t = (2/k)\,t_1$, $\alpha(t) = \alpha(0)\,\alpha_1(t_1)$, $\beta(t) = \alpha(0)\,\beta_1(t_1)$, $R_1 = \pi R/k$ and the dot denotes differentiation with respect to t_1. The equations were integrated numerically for a range of values of the Reynolds number with initial conditions $\alpha_1(0) = 1$ and $\beta_1(0) = 0$.

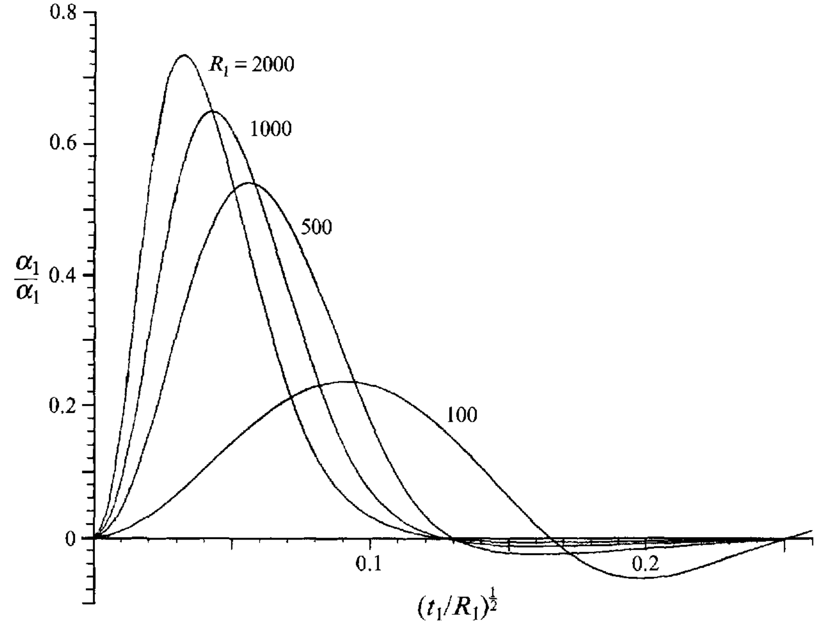

FIGURE 2. Growth of waves on a Rayleigh layer. Amplification rate plotted against $(t_1/R_1)^{\frac{1}{2}}$.

The integrations were carried out up to $t_1 = 0.15R_1$. The solutions are displayed in figures 2 and 3. Figure 2 shows a plot of the amplification rate $\dot{\alpha}_1/\alpha_1$ plotted against $(t_1/R_1)^{\frac{1}{2}} = \frac{1}{4}k\delta_2$ for $R_1 = 100$, 500, 1000 and 2000. The amplification rate reaches a maximum at $(t_1/R_1)^{\frac{1}{2}} = (t_1/R_1)^{\frac{1}{2}}_{max}$ and is suppressed at $(t_1/R_1)^{\frac{1}{2}} = (t_1/R_1)^{\frac{1}{2}}_{c}$, both of which times are strictly quite outside the range of validity of the governing equations. However, it is interesting to note the dependence of $(t_1/R_1)^{\frac{1}{2}}_{max}$ on R_1, displayed in figure 3 (bottom curve): $(t_1/R_1)^{\frac{1}{2}}_{max}$ decreases with increase in R_1 and at $R_1 = 2000$ has a value 0.031. This implies that in a high-Reynolds-number flow, waves of wavelength λ on a Rayleigh layer grow fastest when $t = 0.000\,15\lambda^2 R$. Also displayed in figure 3 is the dependence on Reynolds number of $(t_1/R_1)^{\frac{1}{2}}_{c}$ (top curve); at this value of $(t_1/R_1)^{\frac{1}{2}} > 0$, $\dot{\alpha}_1$ vanishes; $(t_1/R_1)^{\frac{1}{2}}_{c}$ decreases with increase in R_1 and at $R_1 = 2000$ has a value 0.125. This implies that waves of wavelength λ stop growing on a Rayleigh layer in a high-Reynolds-number flow when $t = 0.002\lambda^2 R$. Although equations (5.11) are valid for small times, these values provide useful estimates for the maximum growth rate and the cutoff wavelength.

Figure 2 shows that for $(t_1/R_1)^{\frac{1}{2}} > (t_1/R_1)^{\frac{1}{2}}_{c}$, $\dot{\alpha}_1$ vanishes again when $(t_1/R_1)^{\frac{1}{2}} = 0.24$ and $\dot{\alpha}_1 > 0$ for $(t_1/R)^{\frac{1}{2}}$ greater than this value. This can be inferred from (5.11*a*), where the right-hand side vanishes at this value of $(t_1/R_1)^{\frac{1}{2}}$, and is a spurious consequence of the truncation in the expansion in ϵ made in deriving the governing equation of motion of the layer; a similar situation arises for large values of $k\delta_2$ in case (i) above. The effect corresponds to that found by Moore (1978) in the uniform-vorticity case when equations corresponding to (5.9) for that case were used to study the growth of long waves on a straight uniform vortex layer. As Moore points out, the appearance of this spurious effect means that any attempt to numerically integrate the modified integro-differential equation (4.8) will be faced with a difficulty, for even though the value of $(t_1/R_1)^{\frac{1}{2}} = \frac{1}{4}k\delta_2$ at which the spurious growth appears is quite outside the range of validity of (4.12) (or (4.6)), short-wave disturbances, which will be excited in any numerical calculation, will be amplified. Thus special measures need to be taken in a

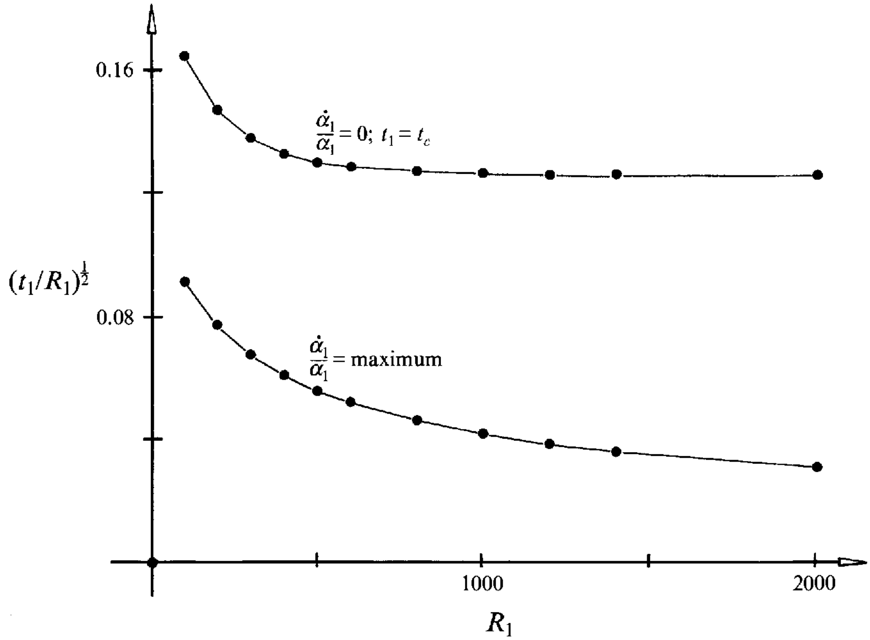

FIGURE 3. Dependence of $(t_1/R_1)^{\frac{1}{2}}_{max}$ and $(t_1/R_1)^{\frac{1}{2}}_c$ on R_1.

numerical scheme to ensure that very short waves are not amplified. A possible remedy to the situation is to obtain a correction of higher order than $O(\epsilon)$ to the governing equation. The matter is pursued in Dhanak (1994) where a higher-order extension to Moore's equation for a thin layer of uniform vorticity is obtained; it is shown that while an expansion to $O(\epsilon^3)$ suppresses growth of short waves over an extended range of wavelengths, very short waves are still amplified and the use of a Padé approximation is suggested. Equations (5.11), however, should give a fairly good description of the growth of long waves on a Rayleigh layer for $(t_1/R_1)^{\frac{1}{2}} \leq (t_1/R_1)^{\frac{1}{2}}_{max}$.

6. Conclusions

An extension to Birkhoff's (1962) equation for the motion of a vortex sheet has been obtained to allow for a non-uniform known distribution of vorticity in a thin layer, instead of a sheet of zero thickness; the equation is valid so long as the thickness of the layer is uniformly small compared with the local curvature. If the flow is taken to be inviscid and the vorticity in the layer to be uniform, the equation reduces to that of Moore (1978); in general, the vorticity satisfies a boundary-layer equation. The extension is used to derive an equation of motion of a diffusing vortex sheet in a viscous fluid. The equation is used to study the growth of long waves on a Rayleigh layer. This study reveals a difficulty in the path of numerical integration of the modified integro-differential equation, for although the equation is strictly valid for consideration of long-wave evolution of the sheet, short waves are bound to arise in any numerical calculation, and it is found that while the growth of a certain range of short-wave disturbances on an evolving vortex layer would be suppressed by allowing for viscous diffusion, very short waves would still be spuriously amplified. The difficulty is akin to that found by Moore in the case of a layer of uniform vorticity. Thus, making an approximate allowance for viscous diffusion of vorticity will not prevent the appearance of short-wave chaotic behaviour on a vortex sheet. Nevertheless, by making use of special smoothing techniques (see Moore 1981) to suppress spurious growth of short-wave disturbances, the equation of motion derived here can be used to numerically follow the long-wave evolution of a diffusing vortex sheet.

The paper is based on work carried out by the author at Imperial College, London with the encouragement of Professor D. W. Moore, FRS. It is a pleasure to thank Professor Moore for suggesting the problem and for his advice during the course of the work reported here.

Appendix. Growth of long waves on a straight non-uniform vortex layer in an inviscid layer

Here we consider the instability of an initially straight and steady vortex layer in a non-viscous fluid to disturbances of wavenumber k, where $k/k_1 \ll 1$ (with k_1 as defined below), using the equations given in §3. The results for the growth rate are compared with those of Drazin & Howard (1962).

The vorticity distribution in the unperturbed layer is taken to be $\omega_0(y)$, where (x, y) denotes the position in a Cartesian coordinate system $\overline{OXY}$ with the centroid line C along $\overline{OX}$ for $t<0$. In view of (3.1), it is assumed that $\omega_0(y) \rightarrow 0$ as $y \rightarrow \pm\infty$ at least as fast as $\exp(-k_1|y|)$. Suppose that the streamwise velocities at $y=\pm\infty$ are $\mp\frac{1}{2}V$. Then the perturbed centroid line can be written,

$$\mathscr{X} = \Gamma V^{-1} + f(\Gamma, t), \tag{A 1}$$

where $|\partial f/\partial \Gamma| \ll V^{-1}$. The integrated vorticity function $\hat{\Delta}$ is taken to be

$$\hat{\Delta} = \Delta_0(y) + \Delta'(\Gamma, y, t), \tag{A 2}$$

where

$$\Delta_0(y) = \int_{-\infty}^{y} \omega_0(y')\,\mathrm{d}y' \quad \text{and} \quad |\Delta'| \ll |\Delta_0|, \quad \text{uniformly in} \quad y.$$

Substituting (A 1) and (A 2) into (3.15) with $\bar{\nu}=0$ and using the vorticity equation, gives, on linearizing,

$$\frac{\partial f^*}{\partial t} = \mathscr{I}_1 + \frac{\mathrm{i}V}{2}\left(\frac{\partial^2 f}{\partial \Gamma^2} - \frac{\partial^2 f^*}{\partial \Gamma^2}\right)\int_{-\infty}^{\infty} \Delta_0(V-\Delta_0)\,\mathrm{d}y - \mathrm{i}M', \tag{A 3}$$

$$\frac{\partial \Delta'}{\partial t} + (\tfrac{1}{2}V^2 - V\Delta_0)\frac{\partial \Delta'}{\partial \Gamma} + \frac{\partial \Delta_0}{\partial y}\left[V\int_{-\infty}^{y}\frac{\partial \Delta'}{\partial \Gamma}\mathrm{d}y_1 + M'\right] = V(y\omega_0 - \Delta_0)\left(\frac{\partial}{\partial t} - \frac{V^2}{2}\frac{\partial}{\partial \Gamma}\right)\mathrm{Re}\left(\frac{\partial f}{\partial \Gamma}\right), \tag{A 4}$$

where $\mathscr{I}_1$ is given by (5.3) and $M' = 2\int_{-\infty}^{\infty}(V-\Delta_0)(\partial\Delta'/\partial\Gamma)\,\mathrm{d}y$.

In order to consider modal disturbances, f and Δ' are chosen to be

$$f(\Gamma, t) = \mathrm{e}^{\sigma t}(a_+\,\mathrm{e}^{\mathrm{i}k\Gamma/V} + a_-\,\mathrm{e}^{-\mathrm{i}k\Gamma/V}), \quad \Delta'(\Gamma, y, t) = \mathrm{e}^{\sigma t}(d_+(y)\,\mathrm{e}^{\mathrm{i}k\Gamma/V} + d_-(y)\,\mathrm{e}^{-\mathrm{i}k\Gamma/V}), \tag{A 5}$$

where $a_\pm$ and $d_\pm$ have complete values, $a_\pm$ being constants.

If (A 5) is substituted in (A 3) and (A 4) and the principal value integral is evaluated by contour integration, terms proportional to $\mathrm{e}^{\mp\mathrm{i}k\Gamma/V}$ can be equated to yield equations for $a_\pm$ and $d_\pm$. It is found that $d_-(y) \equiv d_+^*(y) \equiv d^*(y)$ is a solution. Introducing the notation, consistent with Drazin & Howard,

$$w(y) = \sigma - \mathrm{i}k(\tfrac{1}{2}V - \Delta_0(y)); \quad w_{1,2} \equiv w(\pm\infty) = \sigma \pm \mathrm{i}k\tfrac{1}{2}V \tag{A 6}$$

and writing $\mathscr{K}(k,y) = (w_1 - w)/(w_1 - w_2)$ and $w' \equiv \mathrm{d}w/\mathrm{d}y$, we obtain

$$\sigma a_{\pm}^* = \tfrac{1}{2}(w_1 - w_2)\, a_{\mp} - \tfrac{1}{2}k\left[\int_{-\infty}^{\infty} \mathscr{K}(k,y)(w - w_2)\,\mathrm{d}y\right](a_{\mp} - a_{\pm}^*) + 2k\int_{-\infty}^{\infty} \mathscr{K}(k,y)\, d_{\mp}\,\mathrm{d}y, \tag{A 7}$$

$$wd - w'\left[\int_{-\infty}^{y} d(y_1)\,\mathrm{d}y_1 + 2\int_{-\infty}^{\infty} \mathscr{K}(k,y)\, d\,\mathrm{d}y\right] = \tfrac{1}{2}w_1(w - w_2 - yw')(a_- + a_+^*). \tag{A 8}$$

The solution to the integral equation (A 8) is straightforward and on eliminating d between (A 7) and (A 8) we obtain a pair of linear, homogenous equations in $a_{\pm}$, the solution to which exists provided the determinant of a certain 2×2 matrix is zero. This is so provided

$$\tfrac{1}{2}(w_1^2 + w_2^2) + \tfrac{1}{2}k\int_{-\infty}^{\infty} \frac{(w^2 - w_1^2)(w^2 - w_2^2)}{w^2}\,\mathrm{d}y = O(k^2\delta_2^2) \tag{A 9}$$

which is in agreement with Drazin & Howard (1962).

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